Transmission line modeling of acoustical systems with vibrating walls*

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1 Abstract

The effect of vibrating walls on the radiated sound of wind instruments has often been claimed to be audible by musicians and instrument makers. Many scientists resisted such ideas because comprehensible explanations have been missing. In this paper a theory based on forced vibrations is presented which predicts changes in input impedance and transfer function which are qualitatively comparable and in the same order of magnitude as those having been observed in experiments[3]. Sound pressure induced axisymmetric wall displacements are causing pressure fluctuations due to the oscillating volume. These influences have been added to a typical transmission line model[6] propagating complex pairs of sound pressure and flow through a sequence of cylindrical elements represented by their transmission matrices.

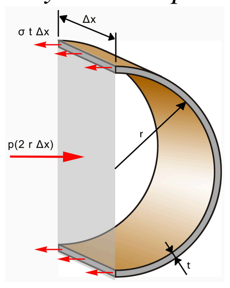
2 Modeling wall vibrations

Elastic strain of the metal wall, which is proportional to the oscillating sound pressure inside the instrument, could provide an explanation for the observed shifts in input impedance and changes in the transfer functions. A first order numerical analysis of such interactions reveals differences in the impedance and transfer function that are of the right order of magnitude and qualitatively similar to the experimental observations.

2.1 Static Case

We begin by calculating the static (purely proportional) case, where the change in diameter of a cylindrical pipe is directly proportional to the applied pressure. In the standard transmission-line model, the instrument is divided into a series of short segments as shown in Fig. 1. Since the wall thickness is small compared with the tube radius, there will not be significant stress in radial direction. Stress in axial direction associated with Poisson's ratio and by forces related to pressure gradients are also neglected. Since the pressure distribution along the instrument axis is smooth and never discontinuous, these assumptions also apply to the dynamic case considered below.

Fig 1 *: Wall forces caused by internal pressure*



The balance of forces inside any short segment of length Δx of a pressurized pipe with internal pressure p, radius r and wall thickness t, requires the pressure induced force $F_\sigma = p\,(2\,r\,\Delta x)$ on any cross-section to be compensated by an equal total force in the walls. The hoop stress σ due to the internal air pressure p can therefore be calculated by

$$\sigma = \frac{p\,r}{t}. \tag{1}$$

From the axially symmetric hoop stress σ the relative change of circumference ε is given by

$$\varepsilon = \frac{\sigma}{E}, \tag{2}$$

where E represents the Young's modulus of the material, which is approximately 100-125 GPa for brass. The relative change in the circumference leads to a larger cross sectional area of the pipe when an internal positive pressure is present. The change in pipe radius Δr over the causative pressure p is then given by

$$\frac{\Delta r}{p} = \frac{r^2}{E\,t}. \tag{3}$$

A quasi-static view, considering frequencies small enough to neglect mass inertia of the wall as well as its internal friction against strain, allows one to define the change in the pipe radius at the maximum positive instantaneous pressure $\hat{p}$ as the amplitude of the oscillations in the wall displacement $\hat{s}$,

$$\hat{s} = \hat{p}\,\frac{r^2}{E\,t}. \tag{4}$$

2.2 Flaring Sections

If the tube is not completely cylindrical but instead flares, as in the bell region of most brass instruments, the situation must be reconsidered. The air pressure displaces the wall in normal direction. Since we are interested in the cross-sectional area of a conical slice, which is described in a coordinate system fixed to the air column, the displacement normal to the wall s_p caused by the pressure force $f_p \approx p\,2\,r\,\pi\,\frac{\Delta x}{\cos(\varphi)}$ is translated into a radial shift $s_r = \frac{s_p}{\cos(\varphi)}$.

The displacement s_p will also be larger than that found in the purely cylindrical case because the direction of the displacement is no longer radial, so less circumferential strain is required for the same amount of displacement. This can be taken into account by another factor of $\cos(\varphi)$, which reduces the effective Young's modulus in that case accordingly.

Eq. 4 can be therefore be rewritten as

$$\hat{s} \approx \hat{p}\,\frac{r^2}{E\,t\,\cos(\varphi)^3}. \tag{5}$$

From the amplitude of the wall displacement we can derive the wall velocity $\hat{v} = \omega\,\hat{s}$ and the parasitic flow $\hat{u}_L = \hat{v}\,2\,r\,\pi\,\Delta x$, which is lost into the vibrating wall.

2.3 Dynamic Case

Any damped mechanical system of first order which is driven by a sinusoidally oscillating external force with amplitude $\hat{F}$ and frequency ω can be described by

$$k\,s(t) + \gamma\,s'(t) + m\,s''(t) = \hat{F}\,e^{\mathrm{i}\,\omega\,t}, \tag{6}$$

where k is the effective spring constant, γ is a damping coefficient and $s(t)$ is the displacement of the effective mass m. The velocity and acceleration are denoted as usual by $s' = \frac{ds}{dt}$ and $s'' = \frac{d^2 s}{dt^2}$ respectively.

It is useful to describe the motion in terms of the resonance frequency $\omega_0 = 2\,\pi\,f_0$, the quality factor Q, and the quasi-static displacement amplitude $\hat{s}$ at very low frequencies. The first two of these quantities can be measured, while $\hat{s}$ can be calculated using eq. 4 or 5.

With these quantities we can formulate Eq. 6 as

$$\frac{s''}{{\omega_0}^2} + \frac{s'}{Q\,\omega_0} + s = \hat{s}\,e^{\mathrm{i}\,\omega\,t}, \tag{7}$$

thus defining the correspondences

$$m = \frac{k}{{\omega_0}^2}, \tag{8}$$

$$\gamma = \frac{k}{Q\,\omega_0}, \tag{9}$$

and

$$\hat{F} = k\,\hat{s}. \tag{10}$$

2.3.1 Amplitude and Phase of Wall Vibrations

The solution for the displacement s is again a harmonically oscillating function $s(t) = \hat{A}(\omega)\,e^{\mathrm{i}\,\omega\,t}$ with an amplitude $\hat{A}(\omega)$, which can be obtained by substituting $s(t)$ and its derivatives $s'(t)$ and $s''(t)$ into Eq. 7. Doing so results in

$$\hat{A}(\omega) = \hat{s}\,\frac{{\omega_0}^2}{{\omega_0}^2 + \mathrm{i}\,\omega\frac{\omega_0}{Q} - \omega^2}. \tag{11}$$

The magnitude of the displacement $\left|\hat{A}(\omega)\right|$ is then given by

$$\left|\hat{A}(\omega)\right| = \hat{s}\,\sqrt{\frac{{\omega_0}^4}{\omega^4 + \left(Q^{-2} - 2\right)\,\omega^2{\omega_0}^2 + {\omega_0}^4}}, \tag{12}$$

with the phase being

$$arg(\hat{A}(\omega)) = \arctan\frac{-\omega\,\omega_0}{Q\,\left(\omega_0^2 - \omega^2\right)}. \tag{13}$$

2.3.2 Critical Frequency of Strain Oscillations

If we concentrate the whole mass of one hoop section and introduce an effective spring constant k for the radial displacement s, we can calculate the critical frequency according to $\omega_0 = \sqrt{\frac{k}{m}}$.

For a conical hoop segment of infinitesimal width Δx the total mass is given by

$$m = 2\,r\,\pi\,\frac{\Delta x}{\cos(\varphi)}\,t\,\rho, \tag{14}$$

with ρ being the mass density of the wall material. Using the definition of the spring constant, $k = \frac{F}{s}$, and substituting the total radial force $F = p\,2\,r\,\pi\,\Delta x$, and s given by eq. 5 results in

$$k = \frac{2\,\pi\,\Delta x\,E\,t\,\cos(\varphi)^3}{r}. \tag{15}$$

This leads to an expression for the critical frequency,

$$\omega_0 = \sqrt{\frac{E\,\cos(\varphi)^4}{r^2\,\rho}} = \frac{\cos(\varphi)^2}{r}\,\sqrt{\frac{E}{\rho}}. \tag{16}$$

Note that the critical frequency does not depend on the wall thickness t, but it depends strongly on the flare angle φ and the radius r.

The radius and flare angle dependencies indicate that there is not one single critical frequency, but a spatially distributed range of critical frequencies. That is, there are regional resonances that are excited at different positions as the driving frequency of the air column changes. Thus the wide range of bore radii and flare angles in the bell of typical brass wind instruments causes strain oscillation resonances over a wide range of frequencies, but local to different parts of the bell.

2.3.3 Thermodynamic Pressure Modulation due to Volume Oscillations

The ideal gas equation $p(t)\,V(t) = R\,T\,n(t)$ relates the number of moles of gas $n(t)$, the volume $V(t)$ and the pressure $p(t)$ at constant temperature T at any time t (R being the universal gas constant). The time varying quantities $p(t)$, $V(t)$ and $n(t)$ are usually derived from constant equilibrium conditions p_0, V_0 and n_0 and small harmonically oscillating magnitudes $\hat{p}\,e^{\mathrm{i}\,\omega\,t}$, $\hat{V}\,e^{\mathrm{i}\,\omega\,t}$ and $\hat{n}\,e^{\mathrm{i}\,\omega\,t}$. The value of $\hat{n}$ can be calculated from the ideal gas equation at constant volume V_0 and constant temperature T by $\hat{n} = \frac{V_0\,\hat{p}}{R\,T}$.

Superimposing such volume oscillations with an amplitude $\hat{V}$, which has a phase shift of ϑ with respect to the pressure in the air column, the gas equation becomes

$$(p_0{+}p_+(t))\,(V_0{+}\hat{V}\,e^{\mathrm{i}\,\omega\,t}\,e^{\mathrm{i}\,\vartheta}) = R\,T\,(n_0{+}\frac{V_0\,\hat{p}}{R\,T}\,e^{\mathrm{i}\,\omega\,t}). \tag{17}$$

Neglecting second order terms and solving for the effective time varying pressure $p_+(t)$ yields

$$p_+(t) = \hat{p}\,e^{\mathrm{i}\,\omega\,t} - \frac{p_0}{V_0}\,\hat{V}\,e^{\mathrm{i}\,\vartheta}\,e^{\mathrm{i}\,\omega\,t}. \tag{18}$$

As expected, the effective pressure is composed of the oscillating pressure, which originally modulated $n(t)$, and an oscillating but phase-shifted additional pressure, which is due to the oscillating volume.

This extra pressure amplitude $\hat{p_V}$, which is caused by the wall vibrations, is given by

$$\hat{p_V} = \frac{p_0}{V_0}\,\hat{V}\,e^{\mathrm{i}\,\vartheta} = \frac{p_0}{r^2\,\pi\,\Delta x}\,(2\,r\,\pi\,\Delta x\,\hat{s})\,e^{\mathrm{i}\,\vartheta} = \frac{2\,p_0}{r}\,\hat{s}\,e^{\mathrm{i}\,\vartheta}. \tag{19}$$

2.4 Impedance and Pressure Transfer Function

In one-dimensional transmission line theory complex wave quantities p and u are propagated through sections of arbitrary acoustical ducts according to

$$\begin{aligned} p_1 &= a\,p_2 + b\,u_2 \\ u_1 &= c\,p_2 + d\,u_2, \end{aligned} \tag{20}$$

a, b, c, d being complex frequency-dependent elements of the propagation matrix A which for lossless cylindrical elements is given by[6]

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(k\,L) & \mathrm{i}\,R_0\,\sin(k\,L) \\ \frac{\mathrm{i}}{R_0}\,\sin(k\,L) & \cos(k\,L) \end{pmatrix}, \tag{21}$$

with L being the length of the cylindrical section, the wave number $k = \frac{\omega}{v}$ and the characteristic impedance $R_0 = \frac{\rho_a\,v}{S}$. As usual, v is the speed of sound, ρ_a is the density of air and ω is the angular frequency. The proportionality of c and the inverse proportionality of b to the cross-sectional area S also hold in the lossy as well as in the conical case.

If the resulting pressure p_1 is decreased by the amount $p_V = k_V\,p_1$ caused by the oscillating volume, then a corrected left side pressure p_1^* is obtained. This correction results from a corrected matrix element b^* because the matrix element b is proportional to the characteristic impedance R_0 and therefore inverse proportional to the effective cross-sectional area. The matrix element c is also proportional to the effective cross-sectional area and needs to be adjusted accordingly. These considerations can be formulated according to

$$\begin{aligned} p_1^* &= p_1\,(1 - k_p - k_V) \\ p_1^* &= a\,p_2 + b^*\,u_2 \\ \frac{c^*}{c} &= \frac{b}{b^*}. \end{aligned} \tag{22}$$

Note that all wave quantities p_i and u_i as well as the coefficients a,b,c,d,k_p and k_V are complex and therefore represent an amplitude or scale factor as well as a relative phase or phase shift. Using eq. 20 and the fact that $Z_2 = \frac{p_2}{u_2}$, we obtain modified matrix elements

$$\begin{aligned} b^* &= b\,(1 - k_p - k_V) - a\,Z_2\,(k_p + k_V) \\ c^* &= c\,\frac{b}{b^*}, \end{aligned} \tag{23}$$

which now take wall vibration effects into account. The acoustic impedance therefore propagates through ducts with vibrating walls according to

$$Z_1 = \frac{b^* + a\,Z_2}{d + c^*\,Z_2}, \tag{24}$$

which allows one to calculate the effective propagation coefficients b^* and c^* during accumulation of all propagation matrices when the accumulation process is started at the known radiation impedance at the open mouth of the bell. These modifications due to wall vibration effects can be interpreted as the static cross-sectional area of an element being slightly increased when a non-rigid wall yields to the air column pressure.

2.5 Theoretical Results

A one-dimensional transmission line simulation using lossy cylindrical and conical elements as proposed by Mapes-Riordan, [6] implemented in the Brass Instrument Analysis System (BIAS), was used to calculate input impedance and mouthpiece-to-bell pressure transfer spectra of the trumpet that was used in the experiments.

Fig 2 *: (a) Predicted transfer function of trumpet with walls free to vibrate (black, solid) and with the walls heavily damped (red, dashed).*
(b) Difference in the theoretical transfer functions.
(c) Predicted input impedance of trumpet with walls free to vibrate (black, solid) and with the vibrations heavily damped (red dashed).
(d) Difference in the theoretical input impedance.

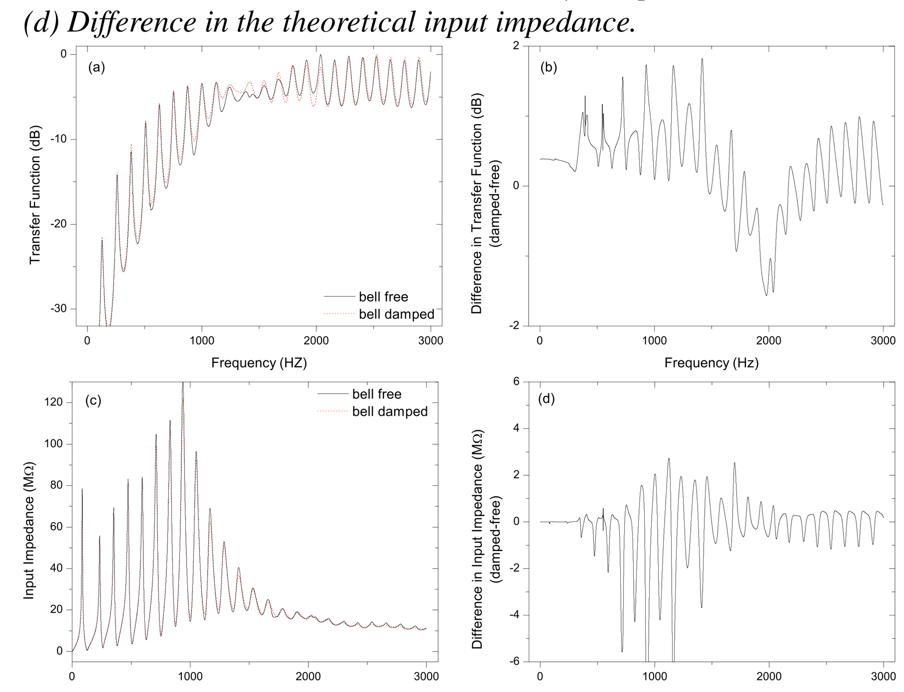


Fig 2(a) shows the predicted pressure transfer spectrum for the case with the vibrations damped and with no damping. The difference between the two cases is shown in Fig. 2(b). The magnitudes of the differences between the two cases predicted by the theory are quite similar to those observed experimentally, as is the qualitative shape of the graph. The impedance spectrum predicted by the model is shown in Fig. 2(c). As is the case with the transfer function, the predicted impedance difference between the damped and free case is similar in magnitude, and a graph of the difference is qualitatively similar in shape to the experimentally derived values.

The similarity between the theoretical and experimental results indicate that a significant portion of the acoustical effects attributable to bell motion during play can be accounted for by assuming an axisymmetric motion of the wall that is caused by the internal air pressure. While these breathing modes occur throughout the instrument, their effect is most pronounced when they occur in the bell section.

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